MATH 211 SPRING 2013 TEST I VERSION A

NAME: KEY

SECTION:

1pt. each

(1) True/False Questions: Here A, B etc. denote sets while $x, y, \cdots$ denote elements, and $\mathcal{P}(A)$ denotes the power set of a set A.

T **F** $\phi \subseteq \{x, y\}$

T **F** $\{x\} = \{\{x\}, \{x\}\}$

T **F** The proposition $(\neg p \rightarrow q) \rightarrow (q \rightarrow p)$ is not a contradiction. $p = F;\ q = F$

T **F** $\{\{\phi\}, \{x, y\}\} \subseteq \mathcal{P}(\{x, y\})$.

T **F** $(A \cap \bar{B}) \cup (\bar{A} \cap B) = U$.

T **F** Given $f : A \rightarrow B$, S and T are subsets of A we have $f(A \cup B) \subseteq f(A) \cup f(B)$.

T **F** $|\mathcal{P}(\{a, b, c\})| = 3^3$.

$A_1 = \{8, 9, 10, 11, 12, 13, 14\}$; $A_2 = \{9, 11, 13,$ …

$A_3 = \{10, 13, 16, \ldots$

$A_4 = \{11, 15, 19, \ldots$

$A_5 = \{12, 17, 22, \ldots$

T **F** For $i = 1, 2, ..$ let the set $A_i = \{7 + i, 7 + 2i, 7 + 3i, ...\}$. Then the set $\bigcap_{i=1}^{\infty} A_i$ has an infinite cardinality. ϕ

T **F** $f : \mathbb{N} \rightarrow \mathbb{N}$ where $f(n) = n^2 + n$ is one-to-one but not onto.

T **F** Given two sets A and B, A can either be an element of B or a subset of B but not both.

T **F** The set $\mathbb{Z} \times \mathbb{Z}$ is countable.

T **F** It is possible to find two countably infinite sets A and B such that $A \cup B$ is uncountable.

38 35

w/o

(2) Show that the proposition $p \to (\neg p \to p)$ is a tautology using 2 pts.

(a) A truth table

p	$\neg p$	$\neg p \to p$	$p \to (\neg p \to p)$
T	F	T	T
F	T	F	T

3 pts. (b) logical equivalences, naming all rules of inference that you use

$\equiv\ p \to (\neg p \to p)$

$\equiv\ \neg p \vee (\neg p \to p)$

$\equiv\ \neg p \vee (\neg(\neg p) \vee p)$

$\equiv\ \neg p \vee (p \vee p)$ double negation rule

$\equiv\ \neg p \vee (p)$ idempotent rule

$\equiv\ \neg p \vee p$

$\equiv\ T$

Example function: examples may vary, but you have to prove your claim!

(3) Find a function f from $\mathbb{N}$ to $\mathbb{N}$ that is neither one to one nor onto. (2 pts)

(1 pt) Let $f(n) = 1$ for all n (constant)

($\frac{1}{2}$pt): f is not 1–1 : both $f(0)$ & $f(1)$ are 1. but $0 \neq 1$.

($\frac{1}{2}$pt): f is not onto : 2 is not an image

(4) Simplify the proposition $(p \oplus q) \leftrightarrow p$ as much as possible. (3 pts)

See next sheet.

#4

$$p \oplus q \leftrightarrow p \equiv (p \oplus q \to p) \wedge (p \to p \oplus q)$$

$$\equiv (\neg(p \oplus q) \vee p) \wedge (\neg p \vee (p \oplus q))$$

$$\equiv (\neg((p \wedge \neg q) \vee (\neg p \wedge q)) \vee p) \wedge (\neg p \vee ((p \wedge \neg q) \vee (\neg p \wedge q)))$$

de Morgan's & associative

$$\equiv ((\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q)) \vee p) \wedge ((\underbrace{\neg p \vee (\neg p \wedge q)}) \vee (p \wedge \neg q))$$

¬p by absorption

de Morgan's & double negation

$$\equiv (((\neg p \vee q) \wedge (p \vee \neg q)) \vee p) \wedge (\neg p \vee (p \wedge \neg q))$$

distributive property

$$\equiv ((\neg p \vee q \vee p) \wedge (p \vee \neg q \vee p)) \wedge ((\neg p \vee p) \wedge (\neg p \vee \neg q))$$

$$\equiv ((T \vee q) \wedge (p \vee \neg q)) \wedge ((T) \wedge (\neg p \vee \neg q))$$

$$\equiv (T \wedge (p \vee \neg q)) \wedge (T \wedge (\neg p \vee \neg q))$$

$$\equiv (p \vee \neg q) \wedge (\neg p \vee \neg q)$$

$$\equiv (p \wedge \neg p) \vee \neg q$$

$$\equiv F \vee \neg q$$

$$\equiv \neg q.$$

2 pts.

(7) Consider a geometric sequence with $a = 3$ and $r = 2$. Find $a_7 + \cdots + a_{14}$ (by applying the formula).

$$a_7 + \cdots + a_{14} = (a_0 + \cdots + a_{14}) - (a_0 + \cdots + a_6)$$

$$= a \cdot \frac{r^{15}-1}{r-1} - a\frac{r^7-1}{r-1} = 3\left(2^{15}-1-2^7+1\right)$$

$$= 3\left(2^{15}-2^7\right).$$

3 pts

(8) Consider the premises: "Every computer science major takes discrete mathematics". "If a student takes discrete mathematics then he receives an overall semester average above 72". "Salwa scored an average below 72". Show that the statement: "Salwa is not a computer science major" is a valid conclusion. Make sure you define appropriate predicates and indicate the names of all rules of inference you use in your proof.

domain : all students

$C(x)$: "x is a C.S. major".

$D(x)$: "x takes discrete".

$A(x)$: "x receives average 72".

1. $\forall x (C(x) \longrightarrow D(x))$ premise
2. $\forall x (D(x) \longrightarrow A(x))$ premise
3. $C(\text{salwa}) \longrightarrow D(\text{salwa})$ univ. inst. & 1
4. $D(\text{salwa}) \longrightarrow A(\text{salwa})$ uni. inst & 2
5. $C(\text{salwa}) \longrightarrow A(\text{salwa})$ (3), (4) & hyp. syll.
6. $\neg A(\text{salwa})$ premise
7. $\neg C(\text{salwa})$ (5), 6 & modus tollens

(5) Does there exist two irrational numbers x and y such that xy is rational? Justify your answer.

2 pts

yes. we Know that $\sqrt{2}$ is irrational so let $x = \sqrt{2}$ & $y = \sqrt{2}$, then $xy = \sqrt{2} \cdot \sqrt{2} = 2$ which is rational

$\Longleftrightarrow$

(6) Prove that an integer n is odd if and only if $n^2 + 11$ is even.

4 pts:

2 pts each direction

let n be odd. then $n = 2K+1$ for some integer K. Then $n^2+11 = (2K+1)^2+11$ $= 4K^2+4K+1+11 = 2(2K^2+2K+6)$ which is even.

Conversely we need to show that if n^2+11 is even then n is odd. we use a contrapositive argument: suppose n is not odd then n is even, so $n=2K$ for some integer K. It follows that $n^2+11 = 4K^2+11 = 2(2K^2+5)+1$. Hence n^2+11 is not even.

(9) Find a function f from $\mathbb{Z}$ to $\mathbb{N}$ that is one to one but not onto.

2 pts.

example:

$$f(z) = \begin{cases} -2z+1 & z \leq 0 \\ 2z & z > 0 \end{cases}$$

Not onto: 0 is not an image of any value in $\mathbb{Z}$.
Suppose it is the image of z.
Case 1: $z \leq 0 \Rightarrow -2z+1 = 0 \rightarrow\leftarrow$ since $-2z \geq 0$ so $-2z+1 > 0$.
Case 2: $z > 0 \Rightarrow f(z) = 2z = 0 \rightarrow\leftarrow$ because $2z > 0$.

f is 1-1: let $f(z_1) = f(z_2)$. If $z_1 \neq z_2$ then: Case 1: both positive so $2z_1 = 2z_2 \rightarrow\leftarrow$. Case 2: both negative: $-2z_1+1 = -2z_2+1 \Rightarrow z_1 = z_2 \rightarrow\leftarrow$. Case 3: $z_1 \geq 0$, $z_2 < 0 \Rightarrow 2z_1 = -2z_2+1 \Rightarrow 2(z_1 - z_2) = 1 \Rightarrow 1$ is even $\rightarrow\leftarrow$. Case 4: $z_1 < 0$, $z_2 \geq 0$: similar to Case 3.

(10) Translate the following to logical notation: "Exactly two cats on campus gave birth this season". (define your predicates, state your domain and use quantifiers as appropriate)

3 pts.

Domain: cats on campus

$C(x)$ = "x gave birth this season"

$$\exists x_1 \exists x_2 \Big((x_1 \neq x_2) \wedge C(x_1) \wedge C(x_2) \wedge \forall x_3 \big(C(x_3) \rightarrow ((x_3 = x_1) \vee (x_3 = x_2)) \big) \Big)$$

[other ways are possible!]